Gauss's Law: applications

Chapter 24

Gauss's Law

The total flux within a closed surface …

… is proportional to the enclosed charge.

$$\Phi = \oint \vec{E} \cdot d\vec{A} = \frac{Q_{enclosed}}{\varepsilon_0}$$

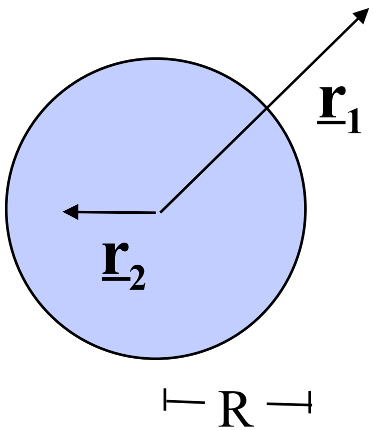
Gauss's Law is always true, but is only useful for certain very simple problems with great symmetry.

Applications of Gauss's Law

We are now going to look at various charged objects and use Gauss's law to find the electric field.

Problem: Sphere of Charge Q

A charge Q is uniformly distributed through a sphere of radius R. What is the electric field as a function of $\underline{\mathbf{r}}$?. Find $\underline{\mathbf{E}}$ at $\underline{\mathbf{r}}_1$ and $\underline{\mathbf{r}}_2$.



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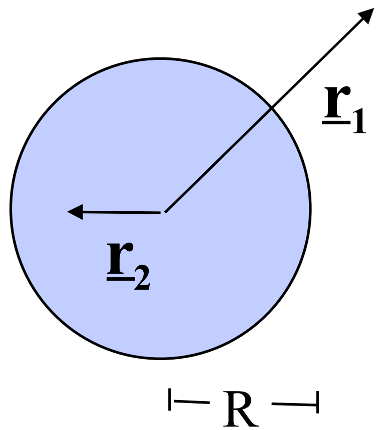

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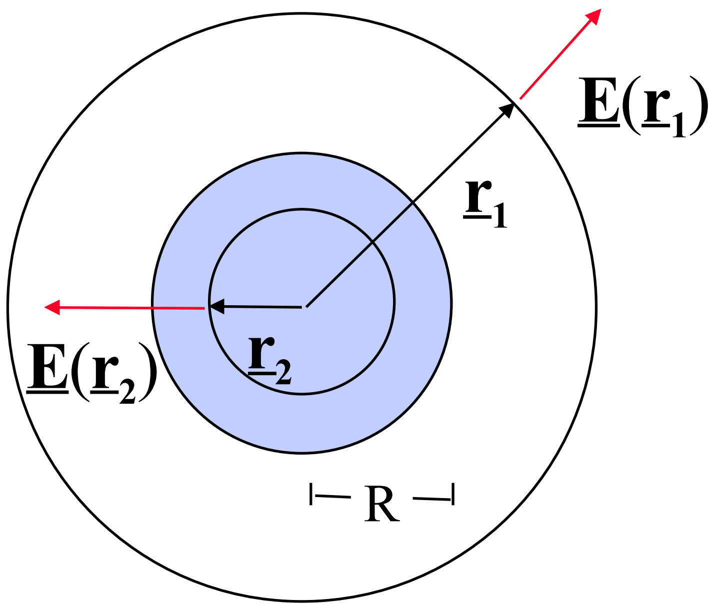
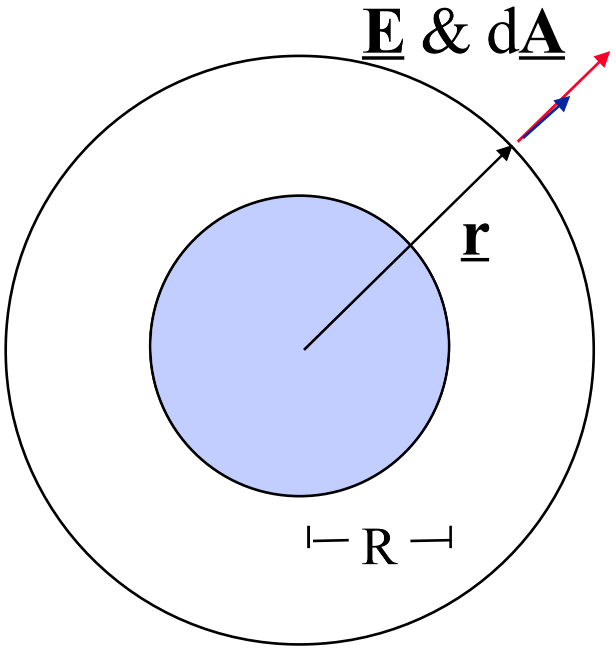


Use symmetry!

This is spherically symmetric. That means that $\underline{\mathbf{E}}(\underline{\mathbf{r}})$ is radially outward, and that all points, at a given radius ($|\underline{\mathbf{r}}|=r$), have the same magnitude of field.

Problem: Sphere of Charge Q

First find **E**(**r**) at a point **outside** the charged sphere. Apply Gauss's law, using as the Gaussian surface the sphere of radius r pictured.



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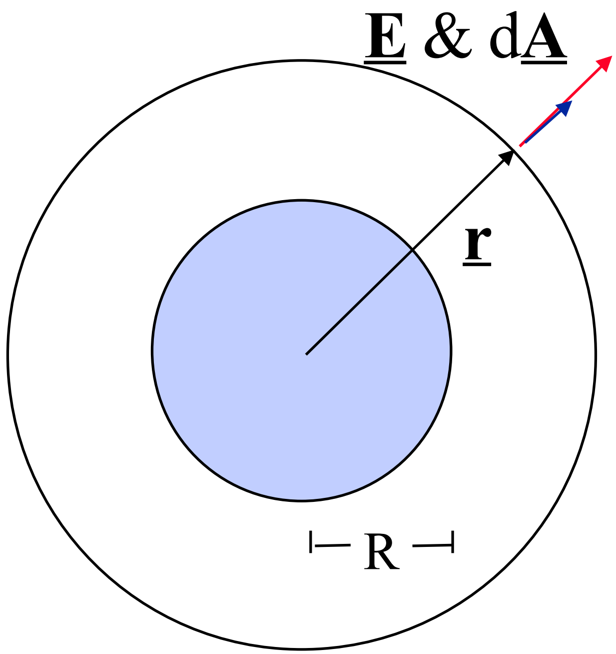

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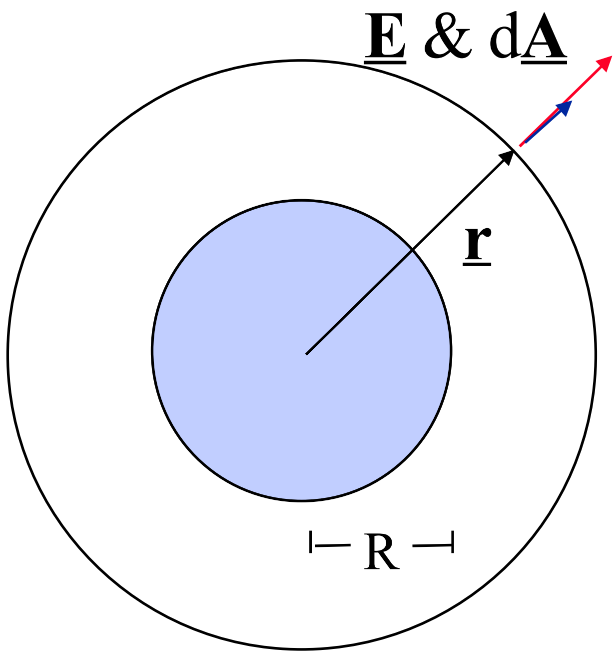

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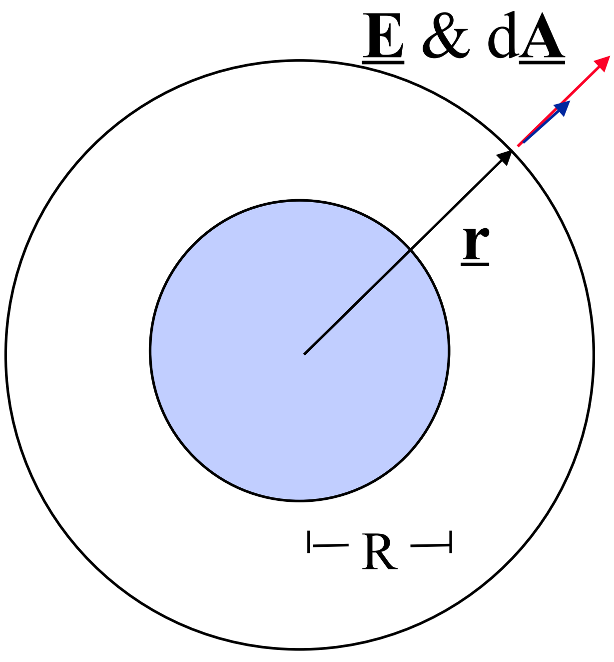

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What is the flux through this surface?

$$\Phi = \oint \vec{\mathbf{E}} \bullet d\vec{\mathbf{A}} = \oint E\,dA$$

$$= E\oint dA = EA = E(4\pi r^2)$$

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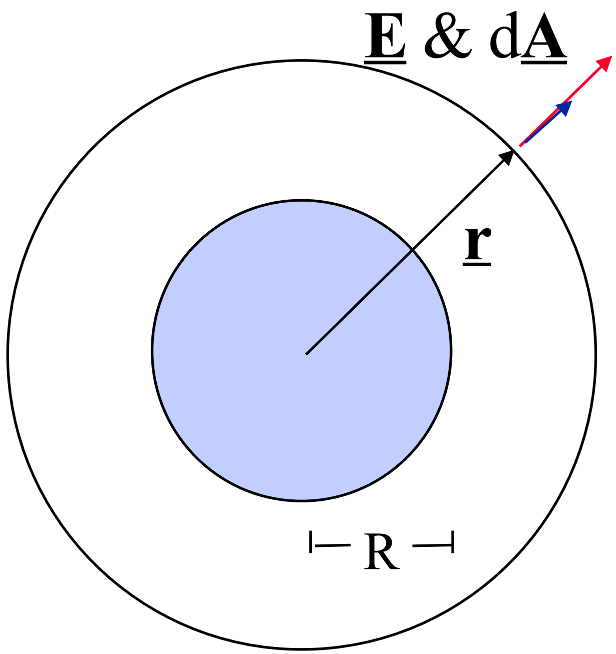

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Gauss $\Rightarrow$ $\Phi = Q/\varepsilon_o$

$$Q/\varepsilon_0 = \Phi = E(4\pi r^2)$$

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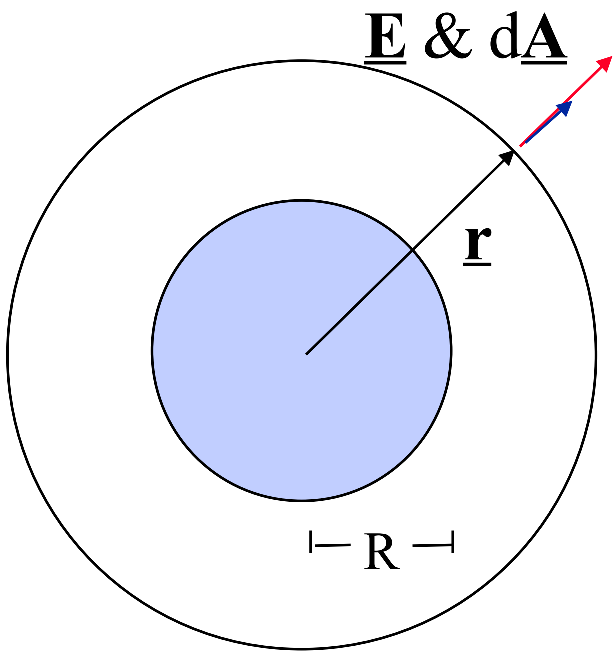

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Gauss: $\Phi = Q/\varepsilon_o$

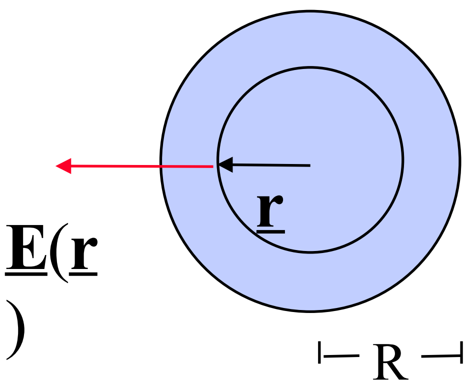
$$Q/\varepsilon_0 = \Phi = E(4\pi r^2)$$

So $\vec{\mathbf{E}}(\vec{\mathbf{r}}) = \dfrac{1}{4\pi\varepsilon_o}\dfrac{Q}{r^2}\hat{\mathbf{r}}$

Exactly as though all the charge were at the origin! (for r>R)

Problem: Sphere of Charge Q

Next find **E**(**r**) at a point **inside** the sphere. Apply Gauss's law, using a little sphere of radius r as a Gaussian surface.



What is the enclosed charge?
That takes a little effort. The little sphere has some fraction of the total charge. What fraction?

That's given by volume ratio: $Q_{enc} = \frac{r^3}{R^3} Q$

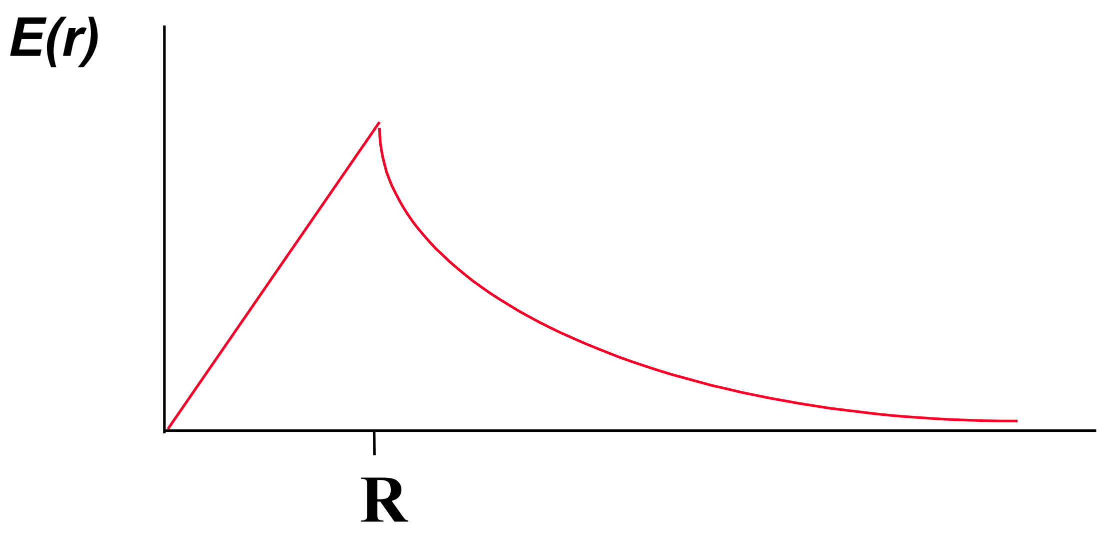
Again the flux is: $\Phi = EA = E(4\pi r^2)$

Setting $\Phi = Q_{enc}/\varepsilon_o$ gives $E = \frac{(r^3/R^3)Q}{4\pi\varepsilon_o r^2}$

For r<R

$$\vec{E}(\vec{r}) = \frac{Q}{4\pi\varepsilon_o R^3} r\hat{r}$$

Problem: Sphere of Charge Q

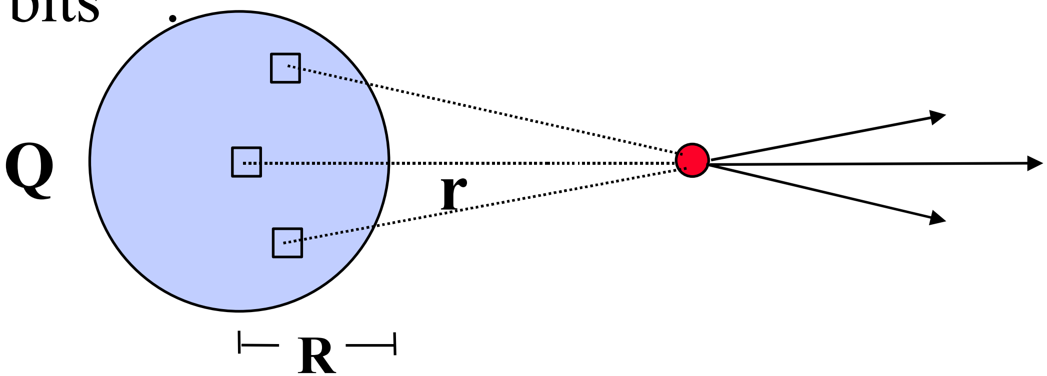


E(r) is proportional to r for r<R
E(r) is proportional to $1/r^2$ for r>R

and E(r) is continuous at R

Problem: Sphere of Charge Q

Look closer at these results. The electric field at [red dot] comes from a sum over the contributions of all the little [blue square] bits.



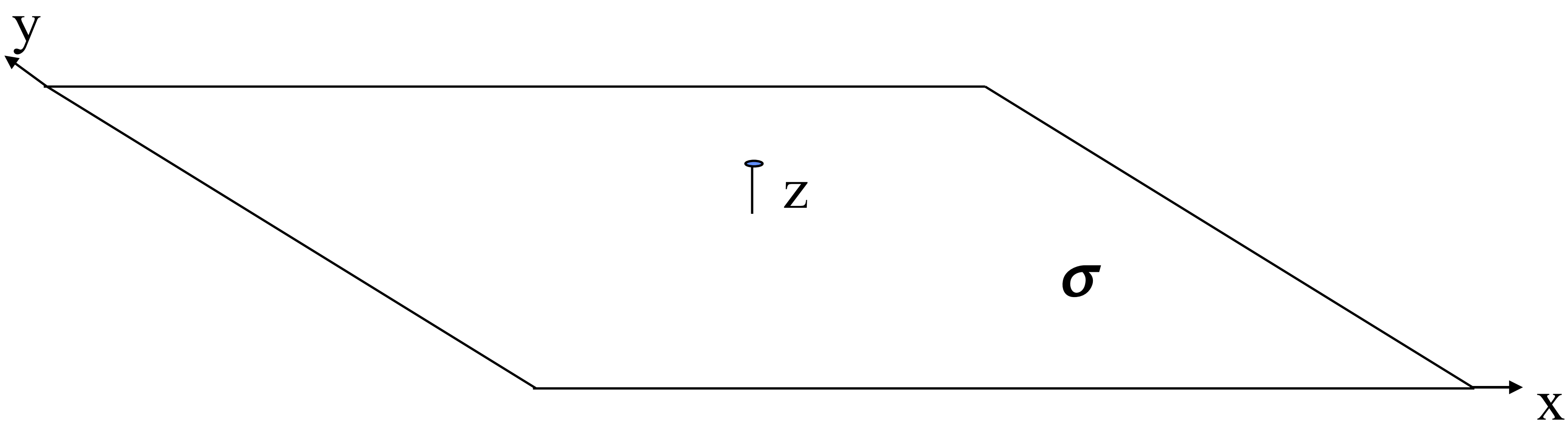
r>R

It's obvious that the net **E** at this point will be horizontal. But the magnitude from each bit is different; and it's completely not obvious that the magnitude E just depends on the distance from the sphere's center to the observation point.

Doing this as a volume integral would be HARD.
Gauss's law is EASY.

Problem: Infinite charged plane

Consider an infinite plane with a constant surface charge density σ (which is some number of Coulombs per square meter).
What is $\underline{\mathbf{E}}$ at a point a distance z above the plane?



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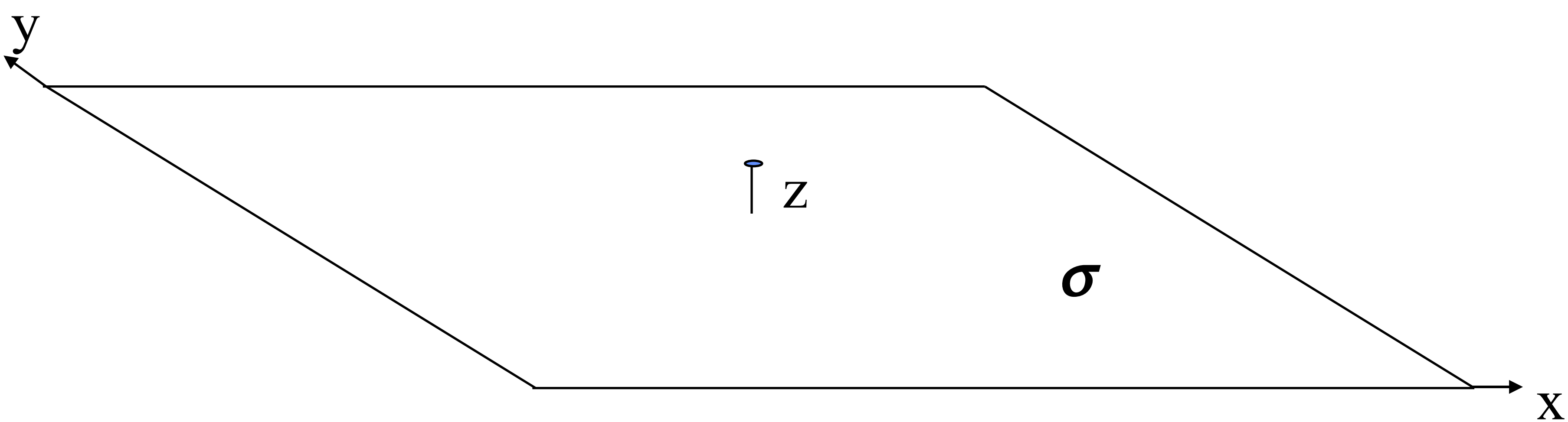

Use symmetry!

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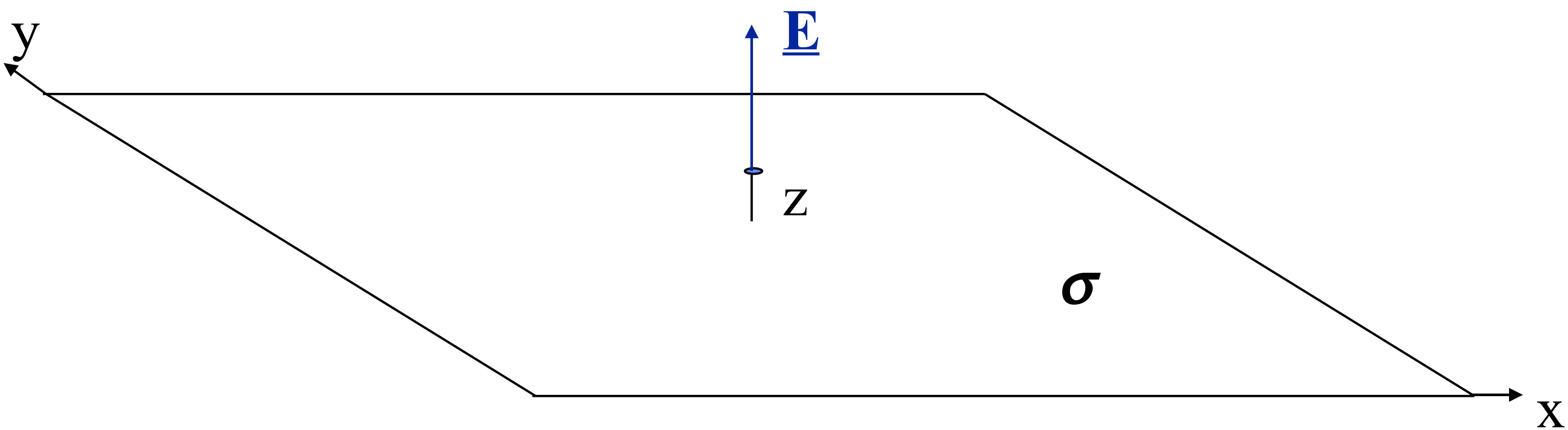
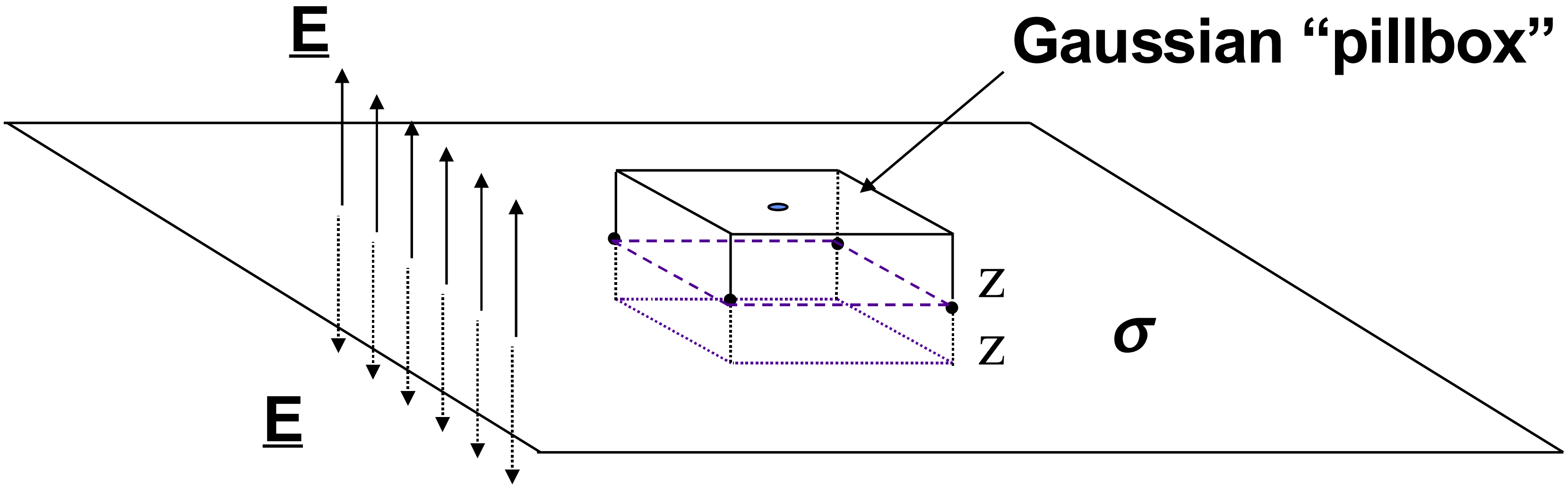


Use symmetry!

The electric field must point straight away from the plane (if σ>0). Maybe the magnitude E depends on z, but the direction is fixed. And **E** is independent of x and y.

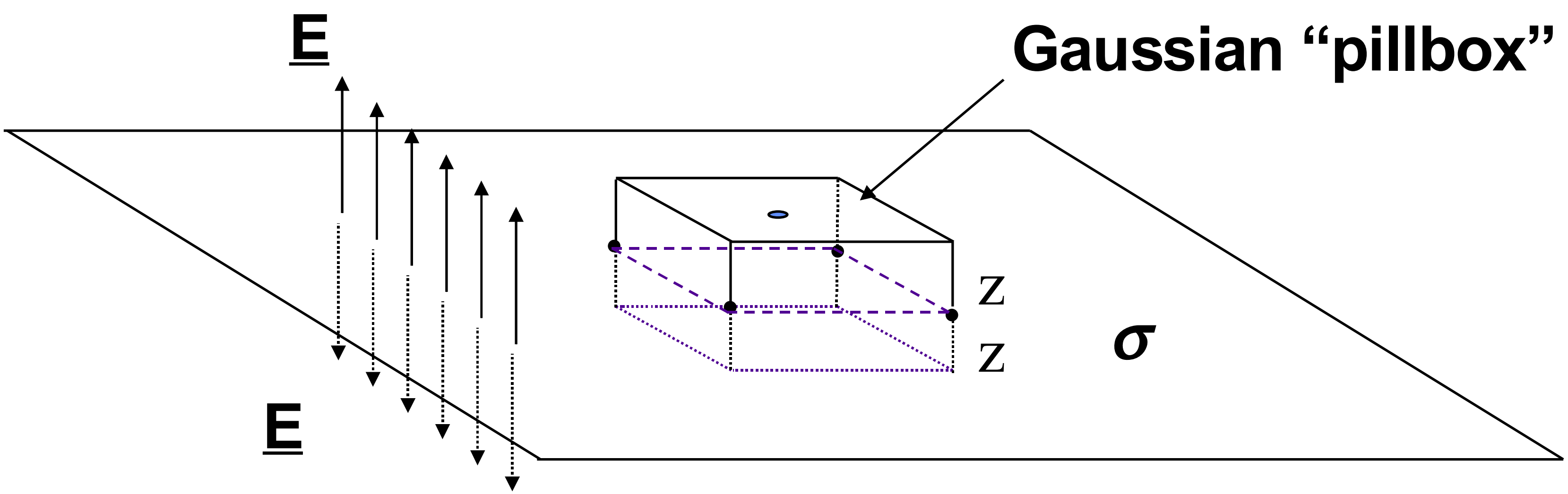
Problem: Infinite charged plane

So choose a Gaussian surface that is a "pillbox", which has its top above the plane, and its bottom below the plane, each a distance z from the plane. That way the observation point lies in the top.



Problem: Infinite charged plane

Let the area of the top and bottom be A.



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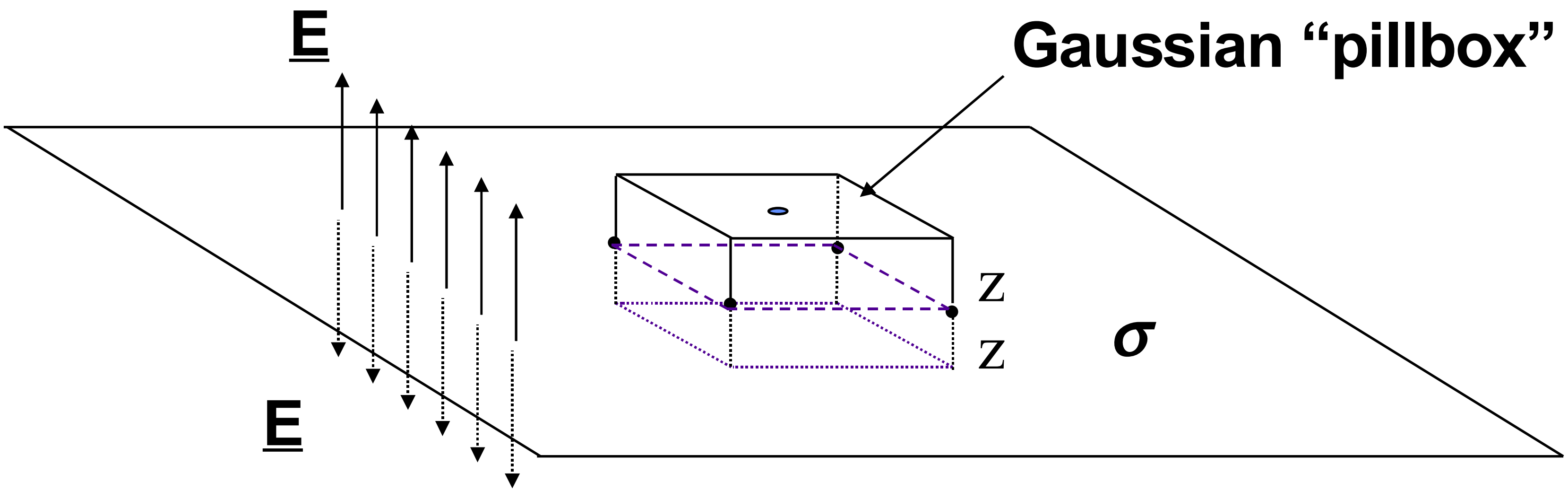

Total charge enclosed by box =

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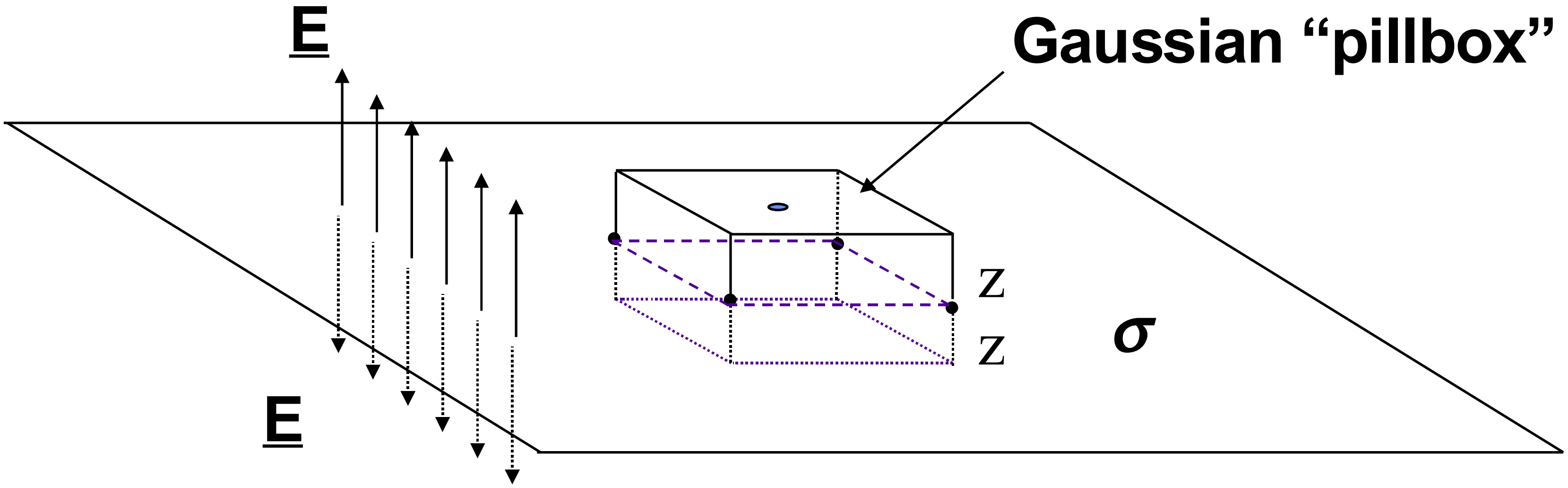

Total charge enclosed by box = Aσ

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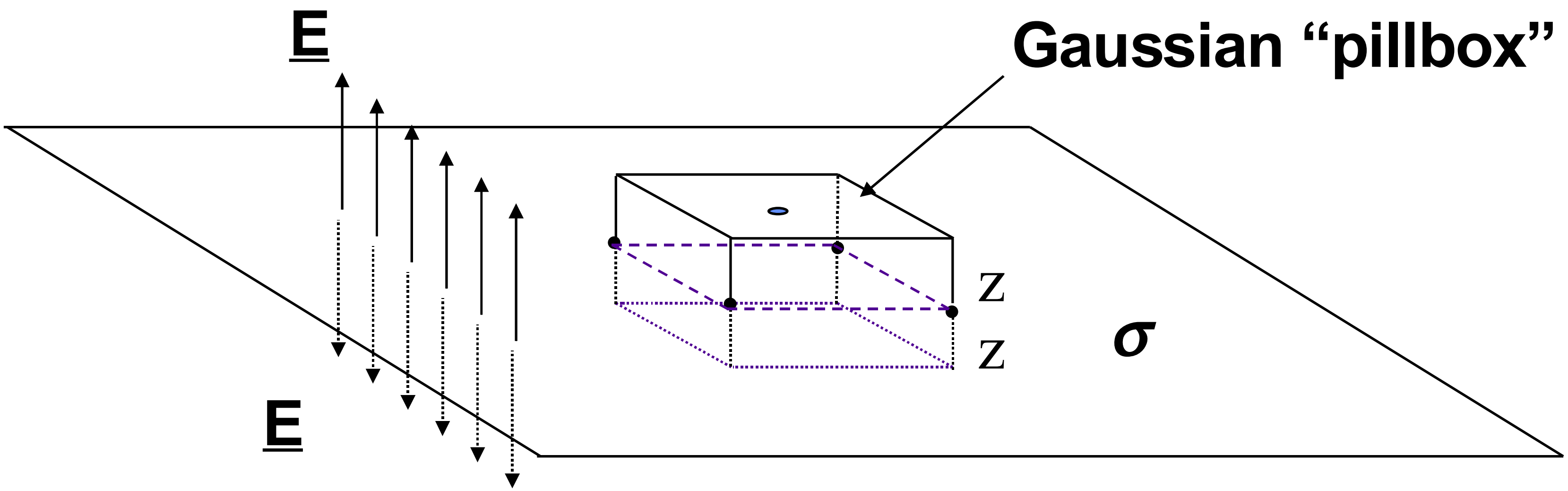

Outward flux through the top:

Problem: Infinite charged plane

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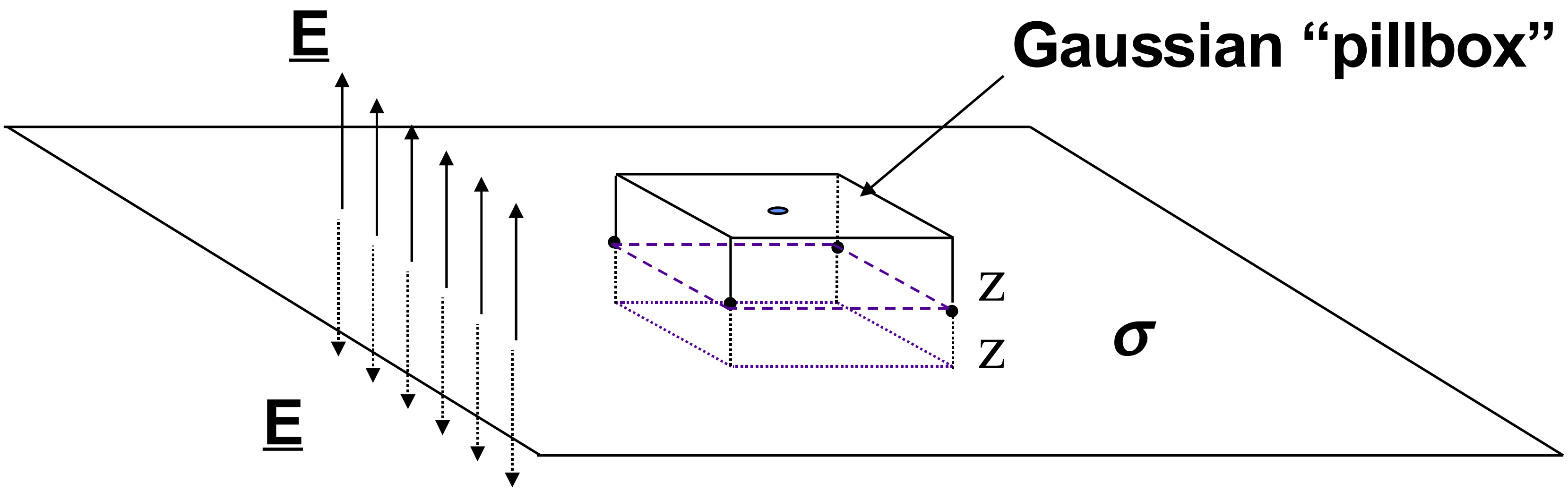

Outward flux through the top: EA

Problem: Infinite charged plane

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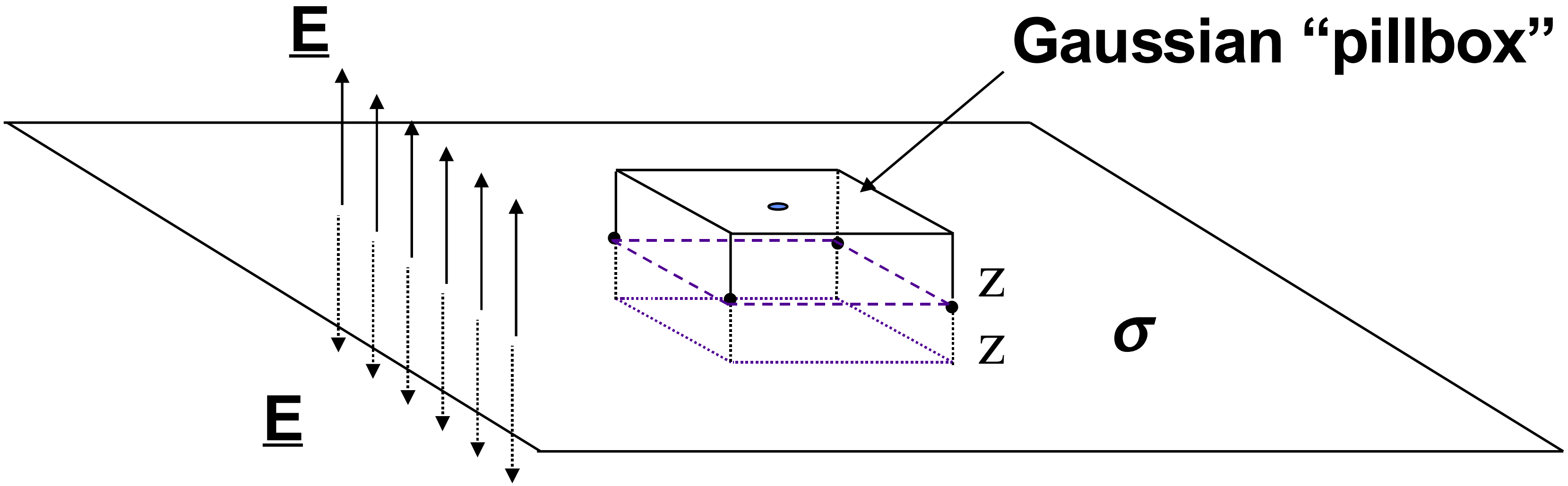

Outward flux through the top: EA

Outward flux through the bottom:

Problem: Infinite charged plane

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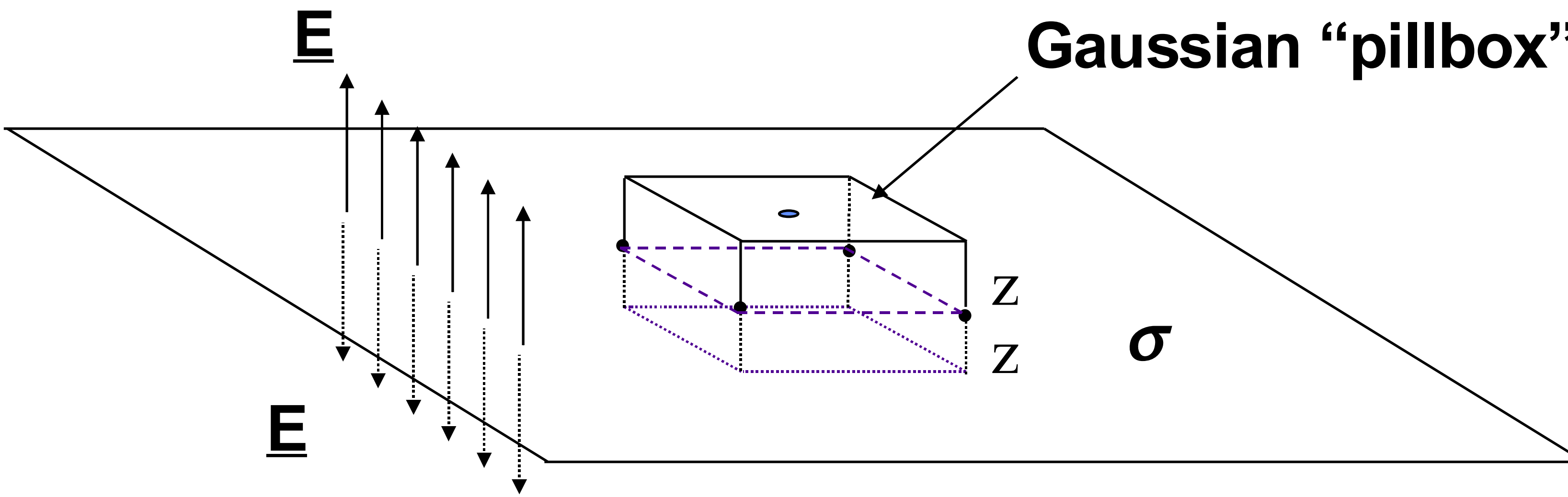

Outward flux through the top: EA

Outward flux through the bottom: EA

Problem: Infinite charged plane

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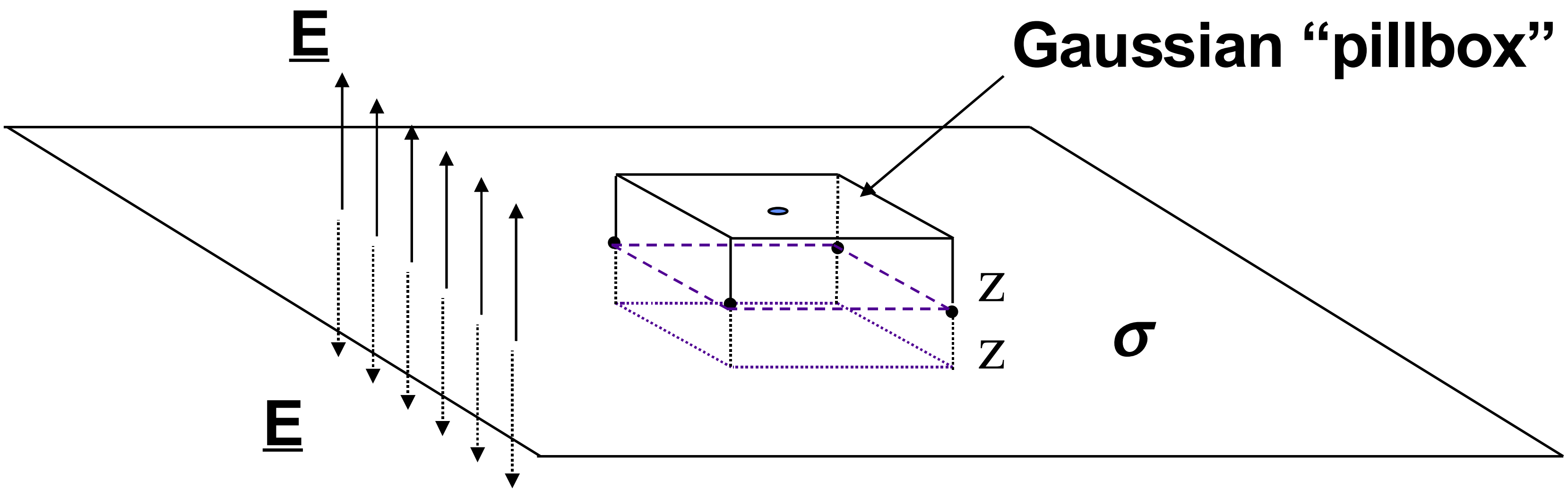

Outward flux through the top: EA
Outward flux through the bottom: EA
Outward flux through the sides:

Problem: Infinite charged plane

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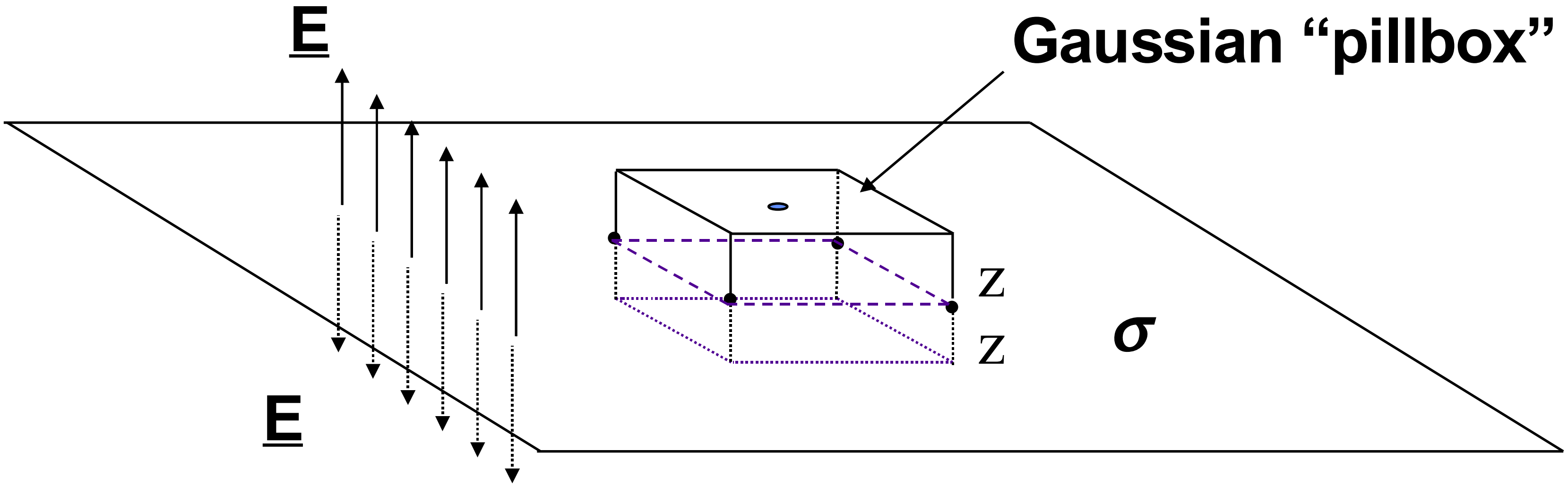

Outward flux through the top: EA

Outward flux through the bottom: EA

Outward flux through the sides: E x (some area) x $\cos(90^0) = 0$

Problem: Infinite charged plane

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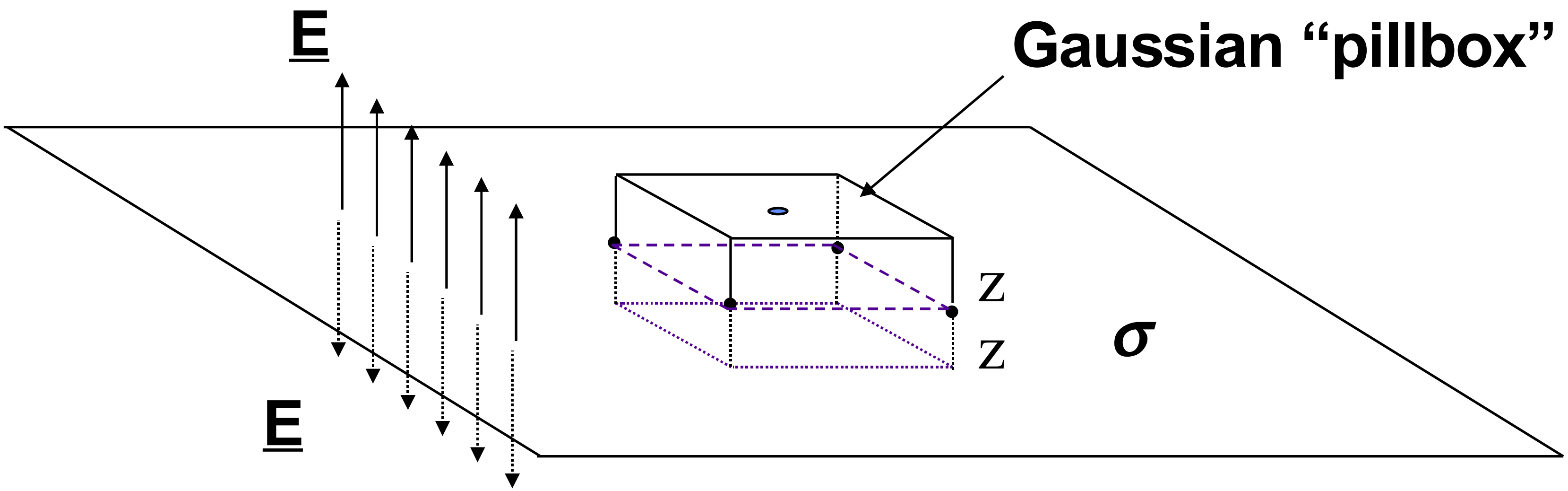

Outward flux through the top: EA

Outward flux through the bottom: EA

Outward flux through the sides: E x (some area) x $\cos(90^0) = 0$

So the total flux is: 2EA

Problem: Infinite charged plane

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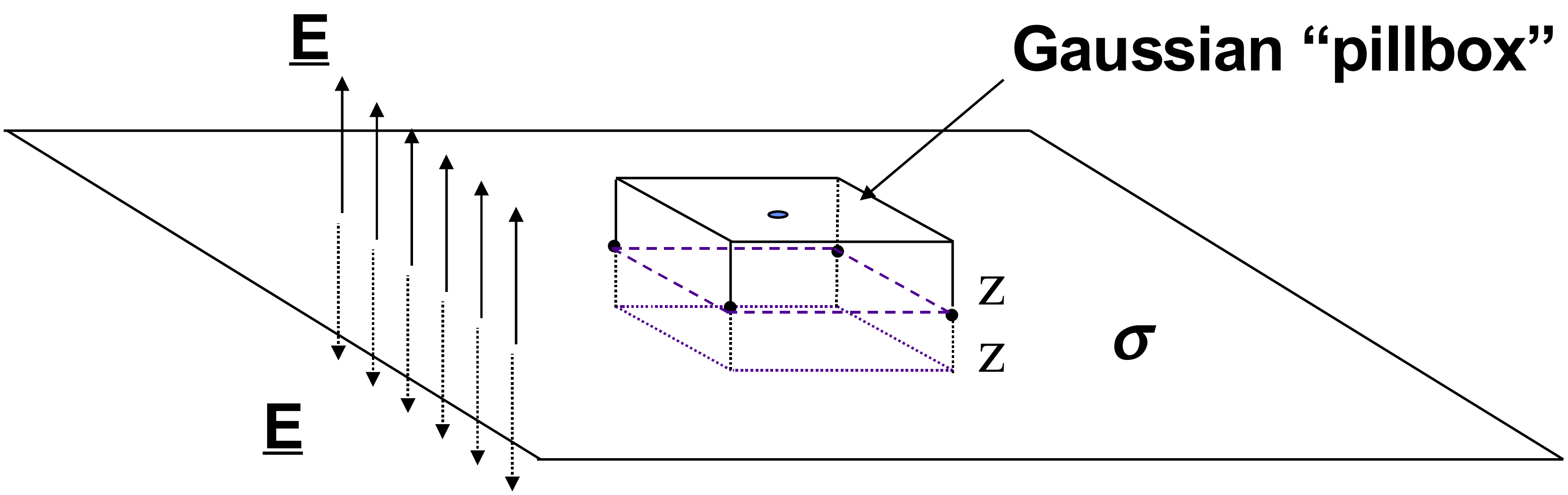


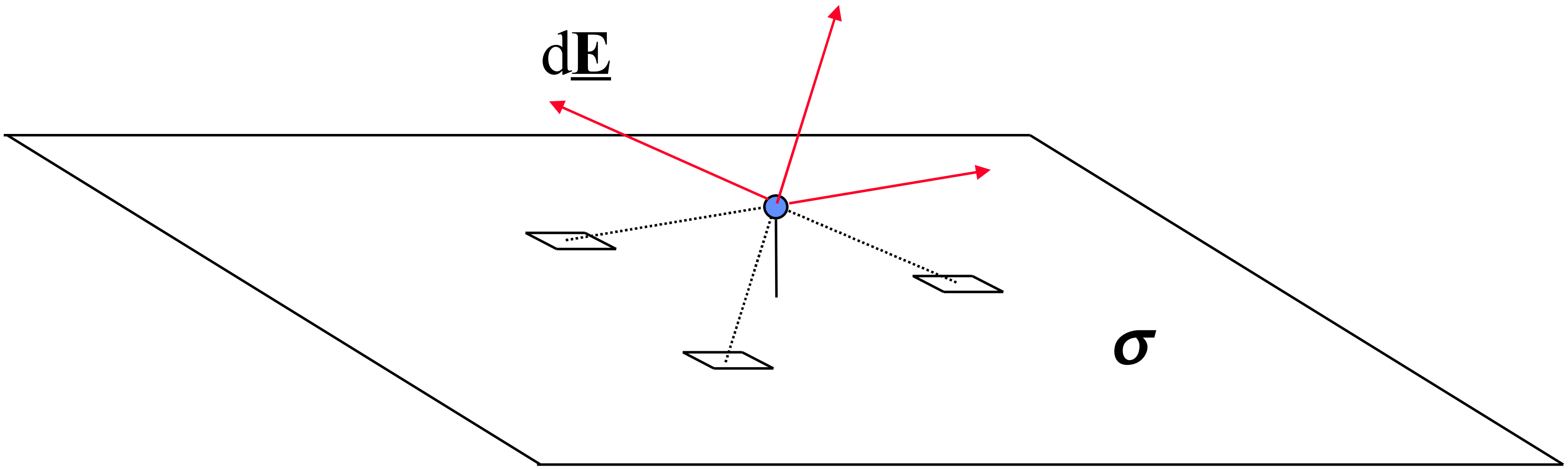
Gauss's law then says that $A\sigma/\varepsilon_0 = 2EA$ so that $\underline{\mathbf{E}} = \sigma/2\varepsilon_0$, outward.

This is constant everywhere in each half-space!

Notice that the area A canceled: this is typical!

Problem: Infinite charged plane

Imagine doing this with an integral over the charge distribution: break the surface into little bits dA …



Doing this as a surface integral would be HARD.
Gauss's law is EASY.

Conductors

- **A conductor is a material in which charges can move relatively freely.**
- **Usually these are metals (Au, Cu, Ag, Al).**
- **In a static condition, the charges placed on a conductor will have moved as far from each other as possible, since they repel each other.**
- **In a static situation, the electric field is zero everywhere inside a conductor.**
- **In a static situation, all the charge resides at the surface of a conductor.**

Conductors

Why is E=0 inside a conductor?

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Why is $\underline{\mathbf{E}}$=0 inside a conductor?

Conductors are full of free electrons, roughly one per cubic Angstrom. These are free to move. If $\underline{\mathbf{E}}$ is nonzero in some region, then the electrons there feel a force $-e\underline{\mathbf{E}}$ and start to move.

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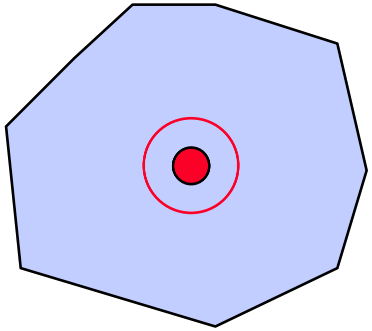
Conductors are full of free electrons, roughly one per cubic Angstrom. These are free to move. If **E** is nonzero in some region, then the electrons there feel a force -e**E** and start to move.

In an electrostatics problem, the electrons adjust their positions until the force on every electron is zero (or else it would move!). That means when equilibrium is reached, **E**=0 everywhere inside a conductor.

Conductors

Because E=0 inside, the inside of a conductor is neutral.

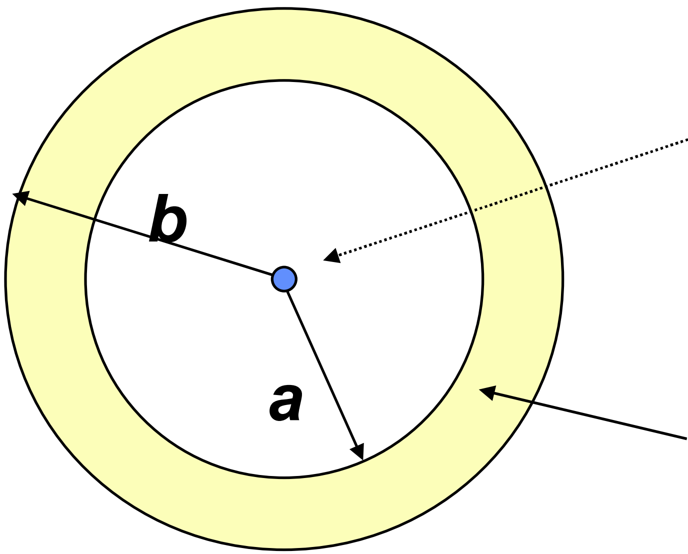


Suppose there is an extra charge ● inside.
Gauss's law for the little spherical surface says there would be a nonzero E nearby.
But there can't be, within a metal!

Consequently the interior of a metal is neutral.
Any excess charge ends up on the surface.

Problem: Charged coaxial cable

This picture is a cross section of an infinitely long line of charge, surrounded by an infinitely long cylindrical conductor. Find **E**.



This represents the line of charge. Say it has a linear charge density of λ (some number of C/m).

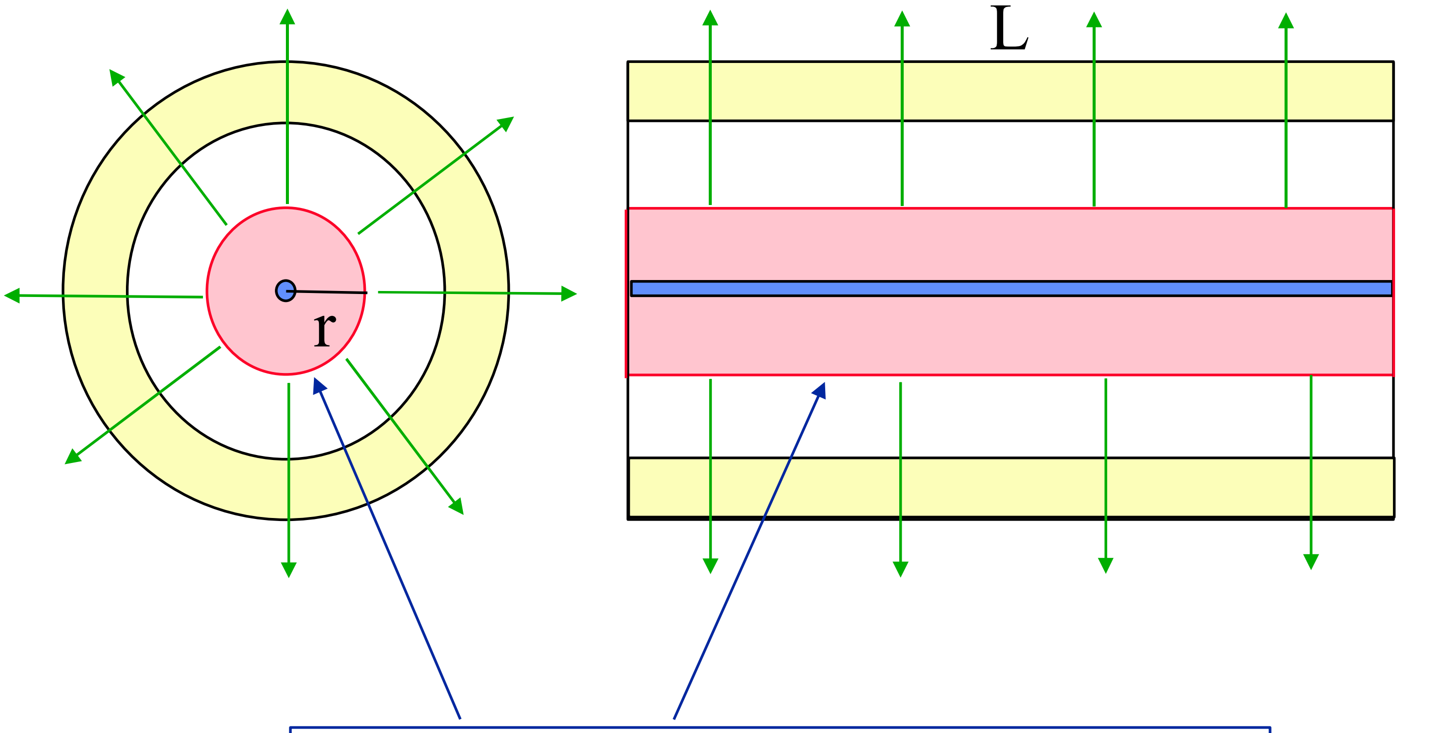
This is the cylindrical conductor. It has inner radius a, and outer radius b.

Use symmetry!

Clearly **E** points straight out, and its amplitude depends only on r.

Problem: Charged coaxial cable

First find E at positions in the space inside the cylinder (r<a).



Choose as a Gaussian surface:
a cylinder of radius r, and length L.

Problem: Charged coaxial cable

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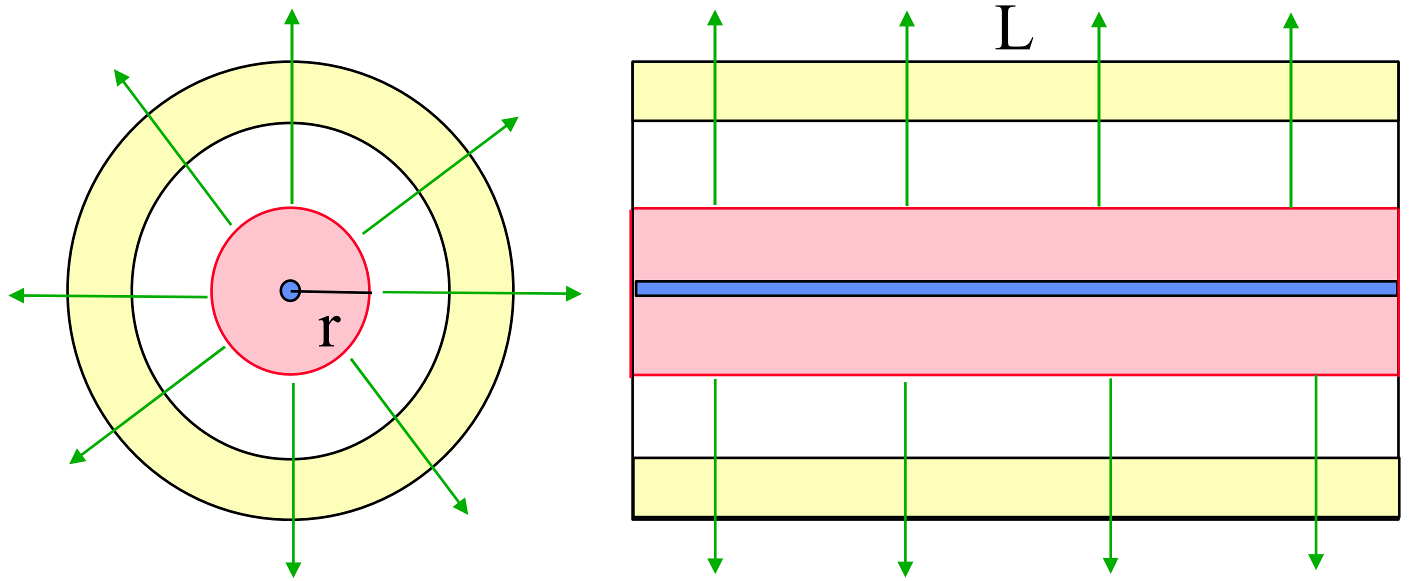


What is the charge enclosed? $\Rightarrow$ λL

What is the flux through the end caps? $\Rightarrow$ zero ($\cos 90^0$)

What is the flux through the curved face? $\Rightarrow$ E x (area) = $E(2\pi rL)$

Total flux = $E(2\pi rL)$

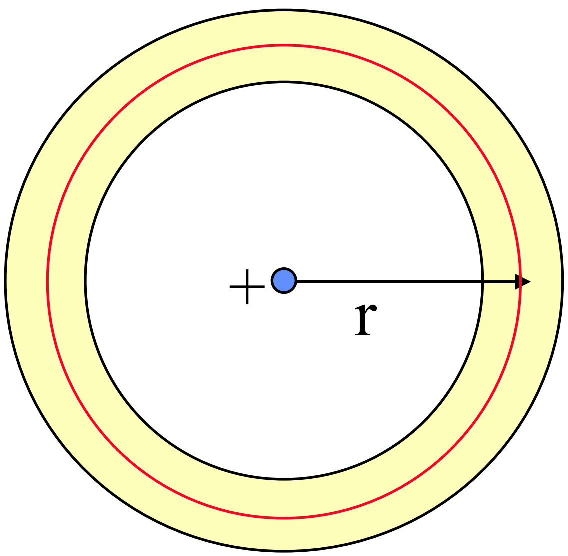
Gauss's law $\Rightarrow$ $E(2\pi rL) = \lambda L/\varepsilon_0$ so **$E(r) = \lambda / 2\pi r\varepsilon_0$**

Problem: Charged coaxial cable

Now find E at positions within the cylinder (a<r<b).

There's no work to do: within a conductor **E**=0.

Still, we can learn something from Gauss's law.

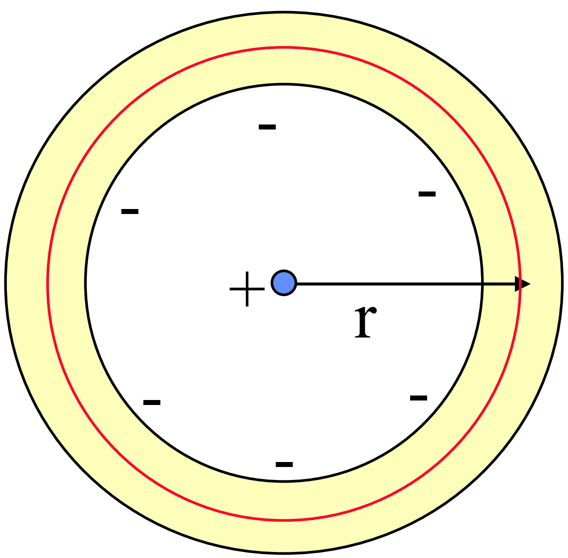


Make the same kind of cylindrical Gaussian surface. Now the curved side is entirely within the conductor, where **E**=0; hence the flux is zero.

Thus the total charge enclosed by this surface must be zero.

Problem: Charged coaxial cable

There must be a net charge per unit length -λ attracted to the inner surface of the metal, so that the total charge enclosed by this Gaussian surface is zero.



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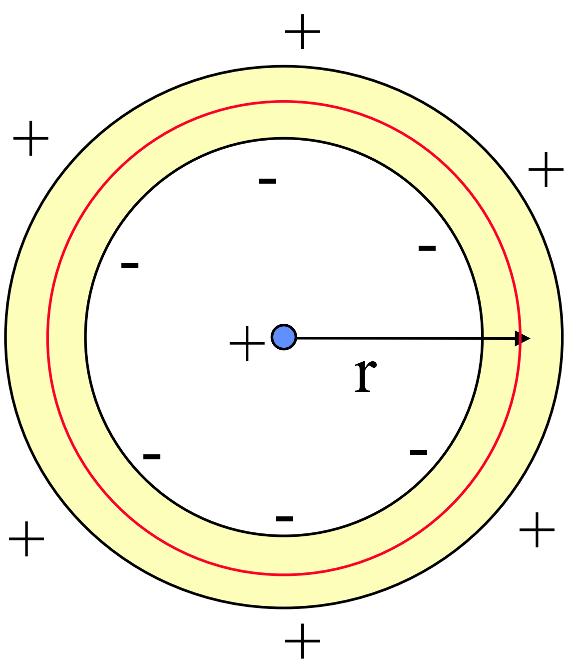

And since the cylinder is neutral, these negative charges must have come from the outer surface. So the outer surface has a charge density per unit length of +λ spread around the outer perimeter.

So what is the field for r>b?